

Deformed Dolan-Grady relations in quantum integrable models

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Abstract

A new hidden symmetry is exhibited in the reflection equation and related quantum integrable models. It is generated by a dual pair of operators $\{A, A^*\} \in \mathcal{A}$ subject to q -deformed Dolan-Grady relations. Using the inverse scattering method, a new family of quantum integrable models is proposed. In the simplest case, the Hamiltonian is linear in the fundamental generators of \mathcal{A} . For general values of q , the corresponding spectral problem is quasi-exactly solvable. Several examples of two-dimensional massive/massless (boundary) integrable models are reconsidered in light of this approach, for which the fundamental generators of \mathcal{A} are constructed explicitly and exact results are obtained. In particular, we exhibit a dynamical Askey-Wilson symmetry algebra in the (boundary) sine-Gordon model and show that asymptotic (boundary) states can be expressed in terms of q -orthogonal polynomials.

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1 Introduction

The exact solution of the planar Ising model in zero magnetic field [1] has provided a considerable source of developments in the theory of exactly solvable systems of statistical mechanics, or quantum field theory in two dimensions. Onsager's successful approach was originally based on the so-called Onsager algebra and its representations. Using algebraic methods, Onsager derived exact results such that the largest and second largest eigenvalues of the transfer matrix of the model. Afterwards, the two-dimensional Ising model was reconsidered using the now famous free fermion (Clifford algebra) techniques [2]. The Onsager algebra itself being not necessary in this latter approach, it probably explains why it assumed only a position of an interesting curiosity in the following years. Despite of this, in the 1980s the Onsager algebra appeared [3, 4] to be closely related with the quantum integrable structure discovered by Dolan and Grady in [5], associated with a class of Hamiltonians of the form

$$H_{DG} = A_0 + \kappa A_1 \quad (1)$$

where κ is a coupling constant and A_0, A_1 are Onsager algebra fundamental generators. More generally, the Onsager (in)finite dimensional algebra in the basis of generators $\{A_m, G_n\}$, $m = 0, \pm 1, \pm 2, \dots$ and $n = 1, 2, \dots$ reads

$$[A_n, A_m] = 4G_{m-n}, \quad [G_m, A_n] = 2A_{n+m} - 2A_{n-m}, \quad [G_m, G_n] = 0. \quad (2)$$

In [5], it was shown that H_{DG} actually belongs to an (in)finite family of mutually commuting conserved quantities provided the so-called Dolan-Grady relations

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1], \quad [A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0] \quad (3)$$

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are satisfied. As pointed out by Dolan and Grady, the power of the formulation (3) relies in the operator statement. Indeed, Hamiltonians of apparently different quantum integrable models can be written as (1) together with (3). For instance, it is the case for the Ising and XY models [5], superintegrable chiral Potts models [6] and some generalizations [7].

Although the Onsager algebra (2) was the most important object in [1], it received less attention in the following years than the star triangle relations (which did not play any essential role in original Onsager's work) did. Actually, the most important progress in the approach of integrable systems was based on the star-triangle relations which originated in [1, 8] and led to the Yang-Baxter equations, the theory of quantum groups, as well as the quantum inverse scattering method. This approach was further extended in [9]: there, it was shown that the Yang-Baxter algebra has to be enlarged with the reflection equation (RE) which arises in various places: factorized scattering theory on the half-line [10], integrable lattice models with non-periodic boundary conditions [9], non-commutative differential geometry on quantum groups. In this generalized framework, the integrability condition of a model follows from the existence of a solution to the reflection equation, the so-called reflection matrix. Indeed, it is the basic object which enters in the definition the quantum transfer matrix [9] denoted $\tau(u)$ below. Using the fact that, by construction, $[\tau(u_1), \tau(u_2)] = 0$ for all u_1, u_2 and expanding in the spectral parameter u , the quantum transfer matrix provides a generating function for an (in)finite number of integrals of motion in the model.

Whereas the inverse scattering method is well understood and provides many non-perturbative results in integrable models (spectrum, scattering amplitudes,...), a formulation based on an (in)finite dimensional algebra of the Onsager-type (2) would be obviously highly desirable: it would provide a new algebraic approach to integrable *massive* models². Consequently, an interesting question is whether such algebraic structure might be actually hidden in the Yang-Baxter/reflection equations. One reason to believe so, is that similarly to the relations (3) the explicit expressions of the R -matrix and K -matrix rely on the (usually non-local) quantum symmetry behind a model (lattice, quantum field theory,...) but not on its explicit microscopic (local) details. As all known models solved using the Onsager algebra's approach can also be analyzed in the framework of the inverse scattering method, it is expected that such a link can be exhibited.

The purpose of this paper is to make a step towards the explicit construction of an (in)finite dimensional symmetry in massive integrable models, along the line initiated in [1, 5]. Indeed, we will show that known integrable models with an underlying $U_{q^{1/2}}(\widehat{sl_2})$ quantum group symmetry, for instance the sine-Gordon quantum field theory or XXZ spin chain, admit an alternative description³ based on a "Hamiltonian" of the form

$$H = A + A^* \quad (4)$$

where q is a deformation parameter and the pair of operators A, A^* (sometimes called a tridiagonal pair [26]) satisfies the " q -deformed" Dolan-Grady relations

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = \rho[A, A^*], \quad [A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = \rho^*[A^*, A]. \quad (5)$$

Here we denote the q -commutator $[A, B]_q = q^{1/2}AB - q^{-1/2}BA$. Notice that these relations can be easily obtained from slightly more general ones called the tridiagonal relations [26] (see also [14, 15]). Also, the special case $q = 1$ in (5) leads to the Dolan-Grady relations (3) [5] (see also [3, 4]). As we are going to see, the "Hamiltonian" (4) differs from the usual *local* one, and is expressed in terms of *nonlocal* objects. Furthermore, it is the first of an infinite set of mutually commuting conserved quantities. On one hand, similarly to the undeformed case (3) the relations (5) lead to an operator statement: they provide the integrability condition for models of the form (4) which does not depend on the dimension of the system or the nature of the space-time manifold. On the other hand, as we shall see the explicit realization/representation of the operators A, A^* will depend on the model under consideration.

²The importance of such infinite dimensional symmetry might be compared to the Virasoro algebra with generators L_n which play a crucial role in the analysis of conformal field theory [11].

³Except if explicitly specified, throughout the paper we assume generic values of q . Among the special cases, one finds the $N = 2$ supersymmetric point in the sine-Gordon model which deserves special attention (for the case with a boundary, see [12, 13]).

The paper is organized as follows. In Section 2, we derive a general solution (so-called K -matrix) of the RE in the spin- $1/2$ representation of $U_{q^{1/2}}(\widehat{sl_2})$. In a first part, assuming that the non-commuting entries $K_{ij}(u)$ of this K -matrix are Laurent polynomials of degree $-2 \leq d \leq 2$ in the spectral parameter u we obtain $K_{ij}(u) \in \mathcal{F}un(u; A, A^*)$ where A, A^* is a “Leonard” pair subject to q -deformed Dolan-Grady relations (5). More general K -matrices, denoted $K^{(n)}(u)$ below, are obtained using a standard “dressing” procedure.

In Section 3, in order to derive independently the previous K -matrix we study the affine quantum group symmetry of the reflection equation. It is shown to be associated with a coaction φ acting on Leonard pairs which preserves the algebraic structure of the reflection equation and generates all $K^{(n)}(u)$ starting from $K^{(0)}(u) \equiv K(u)$. A set of relations for a (universal) K -matrix are also proposed. In some sense, they extend the well-known ones of Drinfeld for the R -matrix. For the spin- $\frac{1}{2}$ representation and using this approach, we explicitly derive $K(u)$. As expected, it agrees with the one obtained by solving directly the reflection equation.

In Section 4, we consider the previous family of solutions $K^{(n)}(u)$ of the RE in the framework of the generalized inverse scattering method for the simplest case $n = 0$. We obtain naturally a generalization of the result of Dolan-Grady [5] for quantum integrable systems. Indeed, the corresponding Hamiltonian (4) is a linear combination of A and A^* satisfying (5), and enjoys a duality property exchanging $A \leftrightarrow A^*$. Its spectral problem is quasi-exactly solvable, except if q is a root of unity in which case it becomes exactly solvable. We give two simple explicit examples of representations for A, A^* satisfying (5). For a single cyclic representation, the form (4) coincides with the Azbel-Hofstadter Hamiltonian on a square lattice in the chiral gauge which spectrum is already known. For a single infinite dimensional representation, one finds Askey-Wilson polynomials as eigenfunctions of (4).

In Section 5, we give some explicit examples of two-dimensional quantum integrable field theory which contain non-local operators satisfying (5). In a first part, we point out that some particular combinations of non-local conserved charges in the sine-Gordon quantum field theory actually satisfy the relations (5), and in some special cases generate the Askey-Wilson algebra. It follows that an infinite set of non-local conserved quantities in *involution* can be obtained. The spectral problem for the simplest one (4) and higher conserved quantities is found to be quasi-exactly solvable and allows us to relate one-particle states to Askey-Wilson polynomials.

In a second part, we apply more general cases $n \neq 0$ of $K^{(n)}(u)$ to two-dimensional boundary quantum integrable models coupled with boundary degrees of freedom located at the boundary, which *must* satisfy (5) in order to ensure integrability. Related integrable models are, for instance, the boundary sine-Gordon model in the massive phase [16, 17, 18] or massless limit [19, 20] coupled with various realizations of A, A^* . It follows that the boundary space of state admits a basis in terms of Askey-Wilson polynomials. Finally, we briefly discuss the weak-strong coupling duality phenomena which relates the bulk algebraic structure (non-local conserved charges) to the boundary degrees of freedom. We find that both operators satisfy the same algebraic relations. Concluding remarks follow in the last Section.

2 Quadratic algebras and deformed Dolan-Grady relations

Among the known examples of quadratic algebraic structures, one finds the Yang-Baxter algebra. For further analysis, let us first recall some known results. This algebra consists of a couple $R(u), L(u)$ where the R -matrix solves the Yang-Baxter equation (YBE) and the L -operator satisfies the quadratic relations (with spectral parameters u, v)

$$R_{\mathcal{V}_0 \mathcal{V}'_0}(u/v) L_{\mathcal{V}_0}(u) \otimes L_{\mathcal{V}'_0}(v) = L_{\mathcal{V}'_0}(v) \otimes L_{\mathcal{V}_0}(u) R_{\mathcal{V}_0 \mathcal{V}'_0}(u/v) \quad (6)$$

where $\mathcal{V}_0, \mathcal{V}'_0$ denote finite dimensional auxiliary space representations. Here, the entries of the L -operator act on a quantum space denoted \mathcal{V} below. If one considers a two-dimensional (spin- $\frac{1}{2}$) representation for \mathcal{V}_0 and \mathcal{V}'_0 , a solution of the YBE is provided by (in general, we will denote $R_{\mathcal{V}_0 \mathcal{V}_0}(u) \equiv R(u)$)

$$R(u) = \sum_{i,j \in \{0,3,\pm\}} \omega_{ij}(u) \sigma_i \otimes \sigma_j, \quad (7)$$

where σ_j are Pauli matrices, $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$ and $\omega_{ij}(u)$ are some combinations of elliptic functions. The corresponding L -operator in (6) can be obtained in analogy of (7). For inequivalent auxiliary and quantum

spaces, it reads

$$L(u) = \sum_{j=\{0,3,\pm\}} \omega_{ij}(u) \sigma_i \otimes S_j \quad (8)$$

where the algebraic properties of the operators $\{S_j\}$ are determined by the Yang-Baxter algebra (6). Here, the operators $\{S_j\}$ act on the quantum space \mathcal{V} . Comparing terms with different dependence on the spectral parameters u, v in (6) one arrives at a set of algebraic relations for $\{S_i\}$, a quadratic algebra which is known as the Sklyanin algebra. However, a simpler solution of the YBE is obtained from the trigonometric limit of $\{\omega_{ij}(u)\}$ in which case the R -matrix takes the form (7) with

$$\begin{aligned} \omega_{00}(u) &= \frac{1}{2}(q^{1/2} + 1)(u - q^{-1/2}u^{-1}) , & \omega_{33}(u) &= \frac{1}{2}(q^{1/2} - 1)(u + q^{-1/2}u^{-1}) , \\ \omega_{+-}(u) &= \omega_{-+}(u) = q^{1/2} - q^{-1/2} . \end{aligned} \quad (9)$$

The corresponding limit of the Sklyanin algebra, also called trigonometric Sklyanin algebra (TSA), is also quadratic. However, it can be shown to coincide with the quantum enveloping algebra $U_{q^{1/2}}(sl_2)$ with the identification

$$S_0 = \frac{q^{s_3/2} + q^{-s_3/2}}{q^{1/4} + q^{-1/4}} , \quad S_3 = \frac{q^{s_3/2} - q^{-s_3/2}}{q^{1/4} - q^{-1/4}}$$

where

$$[s_3, S_{\pm}] = \pm S_{\pm} , \quad \text{and} \quad [S_+, S_-] = \frac{q^{s_3} - q^{-s_3}}{q^{1/2} - q^{-1/2}} , \quad (10)$$

together with the Casimir operator

$$w = q^{1/2}q^{s_3} + q^{-1/2}q^{-s_3} + (q^{1/2} - q^{-1/2})^2 S_- S_+ . \quad (11)$$

An other example of quadratic algebra is provided by the reflection equation (sometimes called boundary Yang-Baxter equation). This equation arises in various context (see for instance [10, 21, 9, 22]). Without spectral parameter, a systematic study of quadratic algebras associated with the RE can be found in [23, 24]. Here we are interested in the spectral parameter dependent form which appeared originally in [10]. For $\mathcal{V}_0 = \mathcal{V}'_0$ it reads

$$R(u/v) (K(u) \otimes \mathbb{I}) R(uv) (\mathbb{I} \otimes K(v)) = (\mathbb{I} \otimes K(v)) R(uv) (K(u) \otimes \mathbb{I}) R(u/v) . \quad (12)$$

Similarly to (8), in the spin- $-\frac{1}{2}$ representation \mathcal{V}_0 we introduce a K -matrix of the form

$$K(u) = \sum_{j \in \{0,3,\pm\}} \sigma_j \otimes \Omega_j(u) . \quad (13)$$

We assume that $\Omega_j(u)$ are Laurent polynomials of degree $-2 \leq d \leq 2$ in the spectral parameter u with coefficients in a (yet unknown) non-commutative associative algebra \mathcal{A} . Replacing (13) in (12), certain coefficients are vanishing which leads to consider

$$\begin{aligned} \Omega_0(u) &= (F + \tilde{G})u/2 - (G + \tilde{F})u^{-1}/2 , & \Omega_+(u) &= Uu^2 + Vu^{-2} + W , \\ \Omega_3(u) &= (F - \tilde{G})u/2 - (G - \tilde{F})u^{-1}/2 , & \Omega_-(u) &= \tilde{U}u^2 + \tilde{V}u^{-2} + \tilde{W} \end{aligned} \quad (14)$$

where $\{U, V, \tilde{U}, \tilde{V}, F, G, \tilde{F}, \tilde{G}, W, \tilde{W}\} \in \mathcal{A}$. Replacing in (12) and after some manipulations we find for instance

$$\begin{aligned} [U, V] &= [U, W] = [V, W] = [\tilde{U}, \tilde{V}] = [\tilde{U}, \tilde{W}] = [\tilde{V}, \tilde{W}] = 0 , \\ [U, \tilde{U}] &= [V, \tilde{V}] = 0 , \\ [U, F] &= [U, G] = [V, F] = [V, G] = [\tilde{U}, F] = [\tilde{U}, G] = [\tilde{V}, F] = [\tilde{V}, G] = 0 , \\ U\tilde{V} &= V\tilde{U} , \quad \tilde{U}V = \tilde{V}U , \end{aligned}$$

and similarly substituting $F \leftrightarrow \tilde{G}, G \leftrightarrow \tilde{F}$. It immediately follows that $\{U, V, \tilde{U}, \tilde{V}\}$ belong to the center of \mathcal{A} . The fixed value of these elements over any quantum space representation \mathcal{V} leads to introduce the linear relations

$$\tilde{U} = c_0 U, \quad \tilde{V} = c_0 V$$

with $c_0 \in \mathcal{C}$. Then, from (12) we also get

$$\begin{aligned} (1 - q^{-1})(U\tilde{W} - c_0 UW) - [F, G] &= 0, & (q - 1)(V\tilde{W} - c_0 VW) - [F, G] &= 0, \\ \tilde{F}U - FV = \tilde{F}\tilde{U} - F\tilde{V} = \tilde{G}V - GU = \tilde{G}\tilde{V} - G\tilde{U} &= 0, \\ [F, G] - [\tilde{F}, \tilde{G}] &= 0. \end{aligned} \tag{15}$$

Assuming that \tilde{W}, W are linearly independent, these relations imply

$$V = q^{-1}U, \quad \tilde{F} = q^{-1}F, \quad \tilde{G} = qG.$$

Let us now turn to bilinear expressions in F, G, W, \tilde{W} following from (12). Using (15), they reduce to

$$\begin{aligned} UF(1 - q^{-2}) + qWG - GW &= 0, & UG(1 - q^2) + WF - qFW &= 0, \\ c_0UF(1 - q^{-2}) + qG\tilde{W} - \tilde{W}G &= 0, & c_0UG(1 - q^2) + F\tilde{W} - q\tilde{W}F &= 0, \\ [W, \tilde{W}] + (q - q^{-1})(q^{-1}F^2 - qG^2) &= 0. \end{aligned} \tag{16}$$

The elements $\{W, \tilde{W}\} \in \mathcal{F}un(F, G)$ may be any polynomials in $\{F, G\}$. However, coefficients of various terms are subjects to the remaining first equation of (15) and eqs. (16). Consistency implies that the allowed combinations are reduced to

$$W = c_1[F, G]_{q^{-1}} + c_2 \quad \text{and} \quad \tilde{W} = -c_0c_1[F, G]_q + c_0c_2, \tag{17}$$

where $\{c_1, c_2\} \in \mathcal{C}$, together with the value of the element (in the center of \mathcal{A}) fixed to $U = -1/c_0c_1q^{-1/2}(q - q^{-1})$. It also follows that $\{F, G\}$ are subject to the trilinear relations

$$\begin{aligned} FG^2 + G^2F - (q + q^{-1})GFG - \frac{q^{-1}}{c_0c_1^2}F &= \frac{c_2}{c_1}(q^{-1/2} - q^{1/2})G, \\ GF^2 + F^2G - (q + q^{-1})FGF - \frac{q}{c_0c_1^2}G &= \frac{c_2}{c_1}(q^{-1/2} - q^{1/2})F. \end{aligned} \tag{18}$$

Notice that these relations are sufficient in order to satisfy the last eq. in (16). Setting

$$F \equiv q^{1/2}A \quad \text{and} \quad G \equiv q^{-1/2}A^*$$

in (18), it is interesting to observe that such kind of trilinear algebraic relations are called the Askey-Wilson relations first introduced by Zhedanov in [51]. They also appear in the context of representation theory associated with tridiagonal algebras (5) (for more details, see [25]). According to [26] an ordered pair A, A^* of linear transformations on a finite dimensional⁴ vector space \mathcal{V}

$$A : \mathcal{V} \rightarrow \mathcal{V} \quad \text{and} \quad A^* : \mathcal{V} \rightarrow \mathcal{V} \tag{19}$$

is a Leonard pair if there exists a basis for \mathcal{V} with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal⁵, and similarly exchanging $A \leftrightarrow A^*$. A Leonard pair is essentially the same thing as a tridiagonal pair for which both operators are diagonalizable with all eigenspaces

⁴It is also possible to find infinite dimensional or cyclic representation \mathcal{V} . Although examples can be found in [26] and below, Leonard pairs in these cases remain to be defined accordingly.

⁵A square matrix X is called tridiagonal whenever each nonzero entry lies on either the diagonal, subdiagonal or the superdiagonal. It is irreducible tridiagonal whenever each entry on the subdiagonal or superdiagonal is nonzero.

of dimension one [[26], Lemma 2.2]. The Leonard pairs are classified in [[25], Theorem 1.9] and [[27], Theorem 5.16]. In [28], it is shown that a Leonard pair A, A^* satisfies

$$\begin{aligned} A^*A^2 + A^2A^* - (q + q^{-1})AA^*A - \gamma(AA^* + A^*A) - \rho A^* &= \gamma^*A^2 + \omega A + \eta I, \\ AA^{*2} + A^{*2}A - (q + q^{-1})A^*AA^* - \gamma^*(AA^* + A^*A) - \rho^*A &= \gamma A^{*2} + \omega A^* + \eta^* I \end{aligned} \quad (20)$$

for a unique sequence of parameters $\gamma, \gamma^*, \rho, \rho^*, \omega, \eta, \eta^*$. To relate the quadratic algebra (12) and (18) to the algebraic relations (20) associated with a Leonard pair A, A^* , we identify⁶ $\gamma = \gamma^* = \eta = \eta^* = 0$ and

$$\rho = \rho^* \equiv \frac{1}{c_0 c_1^2}, \quad \omega \equiv -\frac{c_2}{c_1}(q^{1/2} - q^{-1/2}) \quad \text{and} \quad \gamma = \gamma^* = \eta = \eta^* = 0, \quad (21)$$

where the parameters $c_0, c_1 \neq 0$, c_2 are arbitrary. Combining all previous expressions, we conclude that any solution of the reflection equation (12) of degree $-2 \leq d \leq 2$ in the spectral parameter u - with non-commuting entries - can be written in the form (13) where

$$\begin{aligned} \Omega_0(u) &= (A + A^*)(q^{1/2}u - q^{-1/2}u^{-1})/2, & \Omega_3(u) &= (A - A^*)(q^{1/2}u + q^{-1/2}u^{-1})/2, \\ \Omega_+(u) &= -\frac{q^{1/2}u^2 + q^{-1/2}u^{-2}}{c_0 c_1 (q - q^{-1})} - c_1[A^*, A]_q + c_2, \\ \Omega_-(u) &= -\frac{q^{1/2}u^2 + q^{-1/2}u^{-2}}{c_1(q - q^{-1})} - c_0 c_1[A, A^*]_q + c_0 c_2. \end{aligned} \quad (22)$$

More general solutions of the reflection equation (12) can now be obtained from (13) with (22) using the “dressing” procedure proposed by Sklyanin in [9] that we apply here: Knowing a solution of the reflection equation (12), say $K^{(0)}(u)$, with quantum space $\mathcal{V}^{(0)} \equiv \mathcal{V}$ and given the Lax operator $L_j(u)$ given by (8) with (9), (10) acting on a quantum space \mathcal{V}_j we define a family of “dressed” reflection matrix

$$K^{(n)}(u) \equiv L_n(uk) \cdots L_1(uk) K^{(0)}(u) L_1(uk^{-1}) \cdots L_n(uk^{-1}) \quad (23)$$

acting on the quantum space

$$\mathcal{V}^{(n)} \equiv \bigotimes_{j=1}^n \mathcal{V}_j \otimes \mathcal{V}. \quad (24)$$

Provided $[S_j, a] = 0$ for all $a \in \mathcal{A}$, it is easy to show that $K^{(n)}(u)$ satisfies the RE (12) for any value⁷ of the parameter k . Choosing $K^{(0)} \equiv K(u)$ defined by (13) with (22) and (20), (21), we finally obtain a whole family of solutions of the RE (12).

To conclude, let us mention that finite/infinite dimensional irreducible representations of $K^{(n)}(u)$ in the spin- $\frac{1}{2}$ auxiliary space representation \mathcal{V}_0 are classified according to the tensor product (24). As the tensor product of representations \mathcal{V}_j (associated with $U_{q^{1/2}}(sl_2)$) is well understood, it is important to know all the Leonard pairs as well as their cyclic generalizations. In Section 4 and 5, we will provide explicit examples of such representations, related with integrable systems.

3 Quantum affine reflection algebra

Given any finite dimensional auxiliary space representation $\mathcal{V}_0 = \mathcal{V}'_0$ (for instance the spin- $\frac{1}{2}$ representation) it is well-known that the (four dimensional) matrix solution of the Yang-Baxter equation

$$R(u/v)R(u)R(v) = R(v)R(u)R(u/v) \quad (25)$$

⁶Note that a Leonard pair, say \tilde{A}, \tilde{A}^* , satisfying (20) can be deduced from a different pair A, A^* associated with (20) for $\gamma = \gamma^* = \eta = \eta^* = 0$ and (5) using following substitutions in (20): $A \rightarrow \tilde{A} \equiv A + \alpha I$, $A^* \rightarrow \tilde{A}^* \equiv A^* + \alpha^* I$ and $\gamma \rightarrow -(q^{1/2} - q^{-1/2})^2 \alpha$, $\rho \rightarrow \rho + (q^{1/2} - q^{-1/2})^2 \alpha^2$, $\omega \rightarrow \omega + 2(q^{1/2} - q^{-1/2})^2 \alpha \alpha^*$, $\eta \rightarrow -\rho \alpha^* - \omega \alpha - (q^{1/2} - q^{-1/2})^2 \alpha^2 \alpha^*$ where $\{\alpha, \alpha^*\} \in \mathcal{A}$.

⁷For $k = 1$ and up to an overall spectral parameter dependent factor one has $L^{-1}(u) \sim L(u^{-1})$.

can be derived using the underlying symmetry algebra. For the tensor product of two vector representations, the corresponding R -matrices were derived by Jimbo [29] for all non-exceptional quantum affine Lie algebras. In particular, using the $U_{q^{1/2}}(\widehat{sl_2})$ symmetry in the spin- $\frac{1}{2}$ representation the R -matrix (7) with (9) follows.

An analogous method was initiated in [30] and further developed in [31, 32, 33] in order to derive some K -matrix solutions of the RE (12) using certain coideal subalgebras of quantum Kac-Moody algebras or Yangians [32]. There are, however, known examples of K -matrix derived using different symmetries that do not fit in this framework. For instance, in [18] we introduced an example of “dynamical” extension of coideal subalgebra of $U_{q^{1/2}}(\widehat{sl_2})$. The purpose of this section is to reconsider from a more general point of view the meaning of this extension in the context of $U_{q^{1/2}}(\widehat{sl_2})$ -comodules. In particular, we propose a rather general approach to the construction of a universal K -matrix based on (40).

3.1 Preliminaries

The universal R -matrix associated with the quantum Kac-Moody algebra $U_{q^{1/2}}(\widehat{sl_2})$ was found by Drinfeld [34] using the quantum double construction. This quantum Kac-Moody algebra, denoted for simplicity \mathcal{F} below, is generated by the elements $\{h_j, e_j, f_j\}$, $j \in \{0, 1\}$. They satisfy the commutation relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i/2} - q^{-h_i/2}}{q^{1/2} - q^{-1/2}}$$

together with the q -Serre relations

$$[e_i, [e_i, [e_i, e_j]_q]_{q^{-1}}] = 0, \quad \text{and} \quad [f_i, [f_i, [f_i, f_j]_q]_{q^{-1}}] = 0. \quad (26)$$

The sum $k = h_0 + h_1$ is the central element of the algebra. The Hopf algebra structure is ensured by the existence of a comultiplication $\Delta_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$ and a counit $\mathcal{E}_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{C}$ with

$$\begin{aligned} \Delta_{\mathcal{F}}(e_i) &= e_i \otimes q^{h_i/4} + q^{-h_i/4} \otimes e_i \\ \Delta_{\mathcal{F}}(f_i) &= f_i \otimes q^{h_i/4} + q^{-h_i/4} \otimes f_i, \\ \Delta_{\mathcal{F}}(h_i) &= h_i \otimes \mathbb{I} + \mathbb{I} \otimes h_i \end{aligned} \quad (27)$$

and

$$\mathcal{E}_{\mathcal{F}}(e_i) = \mathcal{E}_{\mathcal{F}}(f_i) = \mathcal{E}_{\mathcal{F}}(h_i) = 0, \quad \mathcal{E}_{\mathcal{F}}(\mathbb{I}) = 1. \quad (28)$$

Furthermore, \mathcal{F} is a coalgebra [35] because

$$(id_{\mathcal{F}} \times \mathcal{E}_{\mathcal{F}}) \circ \Delta_{\mathcal{F}} \cong id_{\mathcal{F}}, \quad (\mathcal{E}_{\mathcal{F}} \times id_{\mathcal{F}}) \cong id_{\mathcal{F}}, \quad (\Delta_{\mathcal{F}} \times id_{\mathcal{F}}) \circ \Delta_{\mathcal{F}} = (id_{\mathcal{F}} \times \Delta_{\mathcal{F}}) \circ \Delta_{\mathcal{F}}.$$

Letting $R \in \mathcal{F} \otimes \mathcal{F}$ denotes the universal R -matrix, as shown in [34] it is the unique invertible element which satisfies the relations:

$$\begin{aligned} \Delta'_{\mathcal{F}}(x) &= R \Delta_{\mathcal{F}}(x) R^{-1} \quad \text{for all } x \in \mathcal{F}, \\ (id_{\mathcal{F}} \times \Delta_{\mathcal{F}})(R) &= R_{13} R_{12}, \quad (\Delta_{\mathcal{F}} \times id_{\mathcal{F}})(R) = R_{13} R_{23} \end{aligned} \quad (29)$$

where $\Delta'_{\mathcal{F}} = \sigma \circ \Delta_{\mathcal{F}}$ and one introduces the permutation map $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in \mathcal{F}$. Here, using the definition $R = \sum_i a_i \otimes b^i$ we denote $R_{12} = \sum_i a_i \otimes b^i \otimes \mathbb{I}$, $R_{13} = \sum_i a_i \otimes \mathbb{I} \otimes b^i$ and $R_{23} = \sum_i \mathbb{I} \otimes a_i \otimes b^i$. In order to obtain a solution of the Yang-Baxter equation (non-linear in the entries of R for some finite dimensional representation), it is sufficient to find $R \in \mathcal{F} \otimes \mathcal{F}$ that satisfies (29). Indeed, from the first equation of (29) one writes

$$R_{12}(\Delta_{\mathcal{F}} \times id_{\mathcal{F}})(x \otimes y) = (\Delta'_{\mathcal{F}} \times id_{\mathcal{F}})(x \otimes y) R_{12} \quad \text{for all } x, y \in \mathcal{F}. \quad (30)$$

Setting $R \equiv x \otimes y$ and using the last relation of (29) one obtains the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} . \quad (31)$$

The universal version of (6) is derived similarly. One denotes the element $L \in \mathcal{F} \otimes \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is a quadratic associative algebra. Actually, suppose that this quadratic algebra is such that there exists an element $L \equiv x \otimes \tilde{a}$ which satisfies (30) substituting $id_{\mathcal{F}} \rightarrow id_{\tilde{\mathcal{A}}}$. Then L satisfies the Yang-Baxter algebra

$$R_{12}L_1L_2 = L_2L_1R_{12} . \quad (32)$$

Also, assuming a Hopf algebra structure for $\tilde{\mathcal{A}}$ with comultiplication $\Delta'_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$, more general solutions $L^{(n)}$ of (32) can be obtained. To construct them, one introduces the n -comultiplication

$$\Delta'^{(n)}_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}} \otimes \cdots \otimes \tilde{\mathcal{A}} \quad , \quad \Delta'^{(n)}_{\tilde{\mathcal{A}}} \equiv (id_{\tilde{\mathcal{A}}} \times \cdots \times id_{\tilde{\mathcal{A}}} \times \Delta'_{\tilde{\mathcal{A}}}) \circ \Delta'^{(n-1)}_{\tilde{\mathcal{A}}} \quad (33)$$

for $n \geq 3$ with $\Delta'^{(2)}_{\tilde{\mathcal{A}}} \equiv \Delta'_{\tilde{\mathcal{A}}}$, and define

$$L^{(n)} \equiv (id_{\mathcal{F}} \times \Delta'^{(n)}_{\tilde{\mathcal{A}}})(L) = L_{\mathbf{n}} \cdots L_1 \quad (34)$$

where indices here refer to the quantum space $\tilde{\mathcal{A}}$. Acting with $(id_{\mathcal{F}} \times \Delta'^{(n)}_{\tilde{\mathcal{A}}})$ on (32) and using the standard comultiplication rules, it is straightforward to show that $L^{(n)}$ is indeed a solution of (32). Obviously, we may have started from $\Delta_{\tilde{\mathcal{A}}}$ instead. In that case, one shows that $\bar{L}^{(n)} \equiv L_1 \cdots L_{\mathbf{n}}$ also solves (32). In other words, the comultiplication $\Delta'_{\tilde{\mathcal{A}}}$ (or similarly $\Delta_{\tilde{\mathcal{A}}}$) preserves the algebraic structure associated with (32).

Later on, we will focus on the spin- $\frac{1}{2}$ representation of $U_{q^{1/2}}(\widehat{sl_2})$ and central element $k = 0$. Denoting $\pi_u^{(\frac{1}{2})}(x)$ the representation with spectral parameter u of the element $x \in \mathcal{F}$, one has

$$(\pi_u^{(\frac{1}{2})} \times \pi_v^{(\frac{1}{2})})[R_{12}] = R(u/v) , \quad (\pi_u^{(\frac{1}{2})} \times id_{\tilde{\mathcal{A}}})[L] = L(u) \quad (35)$$

and more generally

$$(\pi_u^{(\frac{1}{2})} \times id_{\tilde{\mathcal{A}}} \times \cdots \times id_{\tilde{\mathcal{A}}})[L^{(n)}] = L_{\mathbf{n}}(u) \cdots L_1(u) . \quad (36)$$

In particular, the representations (35) are used in (31) and (32) to obtain the Yang-Baxter equation (25) or the Yang-Baxter algebra (6), respectively. In the following subsection, we will proceed by analogy for the reflection equation.

3.2 Comodule algebra and the reflection equation

In this part, we study the algebraic structure related with the spectral parameter dependent reflection equation (12). For $F_q(GL(2))$ i.e. without spectral parameter, part of the following analysis was considered by Kulish and Sklyanin in [23] (see also [24], [36] and [37]). First, let us recall the concept of coaction and comodule necessary in the following analysis. Then, we propose a quite general framework in order to derive a “universal” reflection matrix. The special case of spin- $\frac{1}{2}$ representation is treated in full details.

Definition: [35] *Given a coalgebra $\tilde{\mathcal{F}}$ with comultiplication $\Delta_{\tilde{\mathcal{F}}}$ and counit $\mathcal{E}_{\tilde{\mathcal{F}}}$, \mathcal{A} is called a left $\tilde{\mathcal{F}}$ -comodule if there exists a left coaction map $\zeta : \mathcal{A} \rightarrow \tilde{\mathcal{F}} \otimes \mathcal{A}$ such that*

$$(\Delta_{\tilde{\mathcal{F}}} \times id_{\mathcal{A}}) \circ \zeta = (id_{\tilde{\mathcal{F}}} \times \zeta) \circ \zeta , \quad (\mathcal{E}_{\tilde{\mathcal{F}}} \times id_{\mathcal{A}}) \circ \zeta \cong id_{\mathcal{A}} . \quad (37)$$

Right $\tilde{\mathcal{F}}$ -comodules are defined similarly.

By analogy with the previous subsection, we are now looking for coaction maps φ (and φ') which preserve the algebraic structure associated with the quadratic algebra (12). Similarly to (34), these coaction maps must be such that $K^{(n)}$ is generated from $K^{(0)}$. For a given (here spin- $\frac{1}{2}$) representation of \mathcal{F} , we previously obtained

the solutions $K^{(n)}(u)$ in terms of $K^{(0)}(u)$. For instance, setting $n = 1$ in (23) and replacing $k \rightarrow v$, it leads us to define the coaction maps by

$$\begin{aligned} (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi)(K)] &\equiv L(uv)K(u)L(uv^{-1}) , \\ (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi')(K)] &\equiv L(uv^{-1})K(u)L(uv) . \end{aligned} \quad (38)$$

Here the notation π_v means that we do not specify the dimension of the representation for $\tilde{\mathcal{A}}$. From now on, we identify $\tilde{\mathcal{A}} \equiv \mathcal{F}$. Then, it is important to remember that the L -operators can be written in terms of the universal R -matrix as

$$L(uv^{-1}) = (\pi_u^{(\frac{1}{2})} \times \pi_v)[R_{12}] \quad \text{and} \quad L(uv) = (\pi_u^{(\frac{1}{2})} \times \pi_v)[R_{21}] \quad (39)$$

in order to introduce relations analogous⁸ to (29):

Conjecture: *Let $\Delta_{\mathcal{F}}$ be the comultiplication associated with the coalgebra \mathcal{F} . Let φ (and φ') be a two-sided coaction associated with an associative algebra \mathcal{A} . There exists an element $K \in \mathcal{F} \otimes \mathcal{A}$ which satisfies the relations:*

$$\begin{aligned} (i) \quad & \varphi'(a) = K\varphi(a)K^{-1} \quad \text{for all } a \in \mathcal{A}, \\ (ii) \quad & (\Delta_{\mathcal{F}} \times id_{\mathcal{A}})(K) = (K \otimes \mathbb{I})R_{21}(\mathbb{I} \otimes K) , \quad (\Delta'_{\mathcal{F}} \times id_{\mathcal{A}})(K) = (\mathbb{I} \otimes K)R_{21}(K \otimes \mathbb{I}) , \\ (iii) \quad & (id_{\mathcal{F}} \times \varphi)(K) = R_{21}(K \otimes \mathbb{I})R_{12} , \quad (id_{\mathcal{F}} \times \varphi')(K) = R_{12}(K \otimes \mathbb{I})R_{21} . \end{aligned} \quad (40)$$

Notice that the relation (ii) already appeared in [36]. If such an element K exists, it is then straightforward to show that this element satisfies the reflection equation. There are two possible ways. For instance, from (i) we write

$$(id_{\mathcal{F}} \times \varphi')(x \otimes a)(\mathbb{I} \otimes K) = (\mathbb{I} \otimes K)(id_{\mathcal{F}} \times \varphi)(x \otimes a) \quad \text{for all } x \otimes a \in \mathcal{F} \otimes \mathcal{A} ,$$

or similarly from the first relation of (29) one has

$$R_{12}(\Delta'_{\mathcal{F}} \times id_{\mathcal{A}})(x \otimes a) = (\Delta_{\mathcal{F}} \times id_{\mathcal{A}})(x \otimes a)R_{12} \quad \text{for all } x \otimes a \in \mathcal{F} \otimes \mathcal{A} .$$

Choosing $K \equiv x \otimes a$ and using (iii) or (ii), respectively, in the two equations above we deduce the universal form of the reflection equation

$$R_{12}(K \otimes \mathbb{I})R_{21}(\mathbb{I} \otimes K) = (\mathbb{I} \otimes K)R_{21}(K \otimes \mathbb{I})R_{12} . \quad (41)$$

Similarly to the previous subsection, a generalized n -coaction $\varphi^{(n)}$ can be introduced for $n \geq 2$ with $\varphi^{(1)} \equiv \varphi$. Together with (38), it can be used to generate the family of solutions (23) starting from $K(u) = K^{(0)}(u)$. Consequently, the basic solution (13) with (22) is a fundamental object.

Here, we will not try to construct the universal reflection matrix. Instead, we will consider in details the derivation of $K(u)$ in (13) in the spin- $\frac{1}{2}$ representation of $U_{q^{1/2}}(\widehat{sl_2})$, using (40). Note that a special choice of representation for the associative algebra \mathcal{A} will not be necessary. The spectral parameter dependent form of R , K and L in the spin- $\frac{1}{2}$ representation is given by (35) and

$$(\pi_u^{(\frac{1}{2})} \times \pi_v^{(\frac{1}{2})})[R_{21}] = R(uv) \quad \text{and} \quad (\pi_u^{(\frac{1}{2})} \times id_{\mathcal{A}})[K] = K(u) \quad (42)$$

from which (12) follows. For higher dimensional representation one gets obviously the same form.

Now, we would like to find the explicit expression of the action of φ on the elements of \mathcal{A} . To do that, let us expand (38) in terms of the spectral parameter v . There, we use the explicit form of $K^{(0)}(u)$ given by (13)

⁸However, a rigorous proof of the existence of a unique universal reflection matrix remains to be done.

together with (8). After some straightforward calculations, we find that the $U_{q^{1/2}}(\widehat{sl_2})$ algebra arises explicitly through the evaluation homomorphism $U_{q^{1/2}}(\widehat{sl_2}) \longrightarrow U_{q^{1/2}}(sl_2)$

$$\begin{aligned}\pi_v[e_1] &= vS_+ , & \pi_v[e_0] &= vS_- , \\ \pi_v[f_1] &= v^{-1}S_- , & \pi_v[f_0] &= v^{-1}S_+ , \\ \pi_v[q^{h_1/2}] &= q^{s_3} , & \pi_v[q^{h_0/2}] &= q^{-s_3} .\end{aligned}$$

The expansion of (38) finally gives

$$\begin{aligned}(\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi)(K)]_{11} &= (\pi_{vq^{1/4}} \times id_{\mathcal{A}})[qu^3\zeta(A) - q^{-1}u^{-3}\zeta(A^*) + \zeta'(u, u^{-1}; A, A^*)] , \\ (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi')(K)]_{22} &= (\pi_{vq^{1/4}} \times id_{\mathcal{A}})[qu^3\zeta(A^*) - q^{-1}u^{-3}\zeta(A) + \zeta'(u, u^{-1}; A^*, A)] , \\ (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi)(K)]_{12} &= (\pi_{vq^{1/4}} \times id_{\mathcal{A}})\left[-\frac{qu^4 + q^{-1}u^{-4}}{c_0c_1(q - q^{-1})}\zeta(I) + \zeta''(u^2, u^{-2}; A, A^*)\right] , \\ (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi)(K)]_{21} &= (\pi_{vq^{1/4}} \times id_{\mathcal{A}})\left[-\frac{qu^4 + q^{-1}u^{-4}}{c_1(q - q^{-1})}\zeta(I) + \bar{\zeta}''(u^2, u^{-2}; A^*, A)\right]\end{aligned}\quad (43)$$

where

$$\begin{aligned}\zeta(I) &= \mathbb{I} \otimes I , \\ \zeta(A) &= (c_+e_1 + c_-f_1)q^{h_1/4} \otimes I + q^{h_1/2} \otimes A , \\ \zeta(A^*) &= (c_-e_0 + c_+f_0)q^{h_0/4} \otimes I + q^{h_0/2} \otimes A^* .\end{aligned}\quad (44)$$

The terms ζ' , ζ'' , $\bar{\zeta}''$ do not contribute in the asymptotic $u \rightarrow \infty$ or $u \rightarrow 0$ so we don't write them explicitly here. The parameters c_+ , c_- read

$$c_+ = -1/c_0c_1(q^{1/2} + q^{-1/2}) \quad \text{and} \quad c_- = -1/c_1(q^{1/2} + q^{-1/2}) . \quad (45)$$

It should be stressed that each of these maps ζ, ζ', ζ'' define a left coaction and confirm the fact that \mathcal{A} is a left $U_{q^{1/2}}(\widehat{sl_2})$ -comodule. For instance, it is easy to check using (27), (28) that ζ defined by (44) is consistent with the comultiplication $\Delta'_{\mathcal{F}} = \sigma \circ \Delta_{\mathcal{F}}$, i.e. satisfies (37). Starting from (i) in (40), we can now write (the representation in v -space is not specified yet)

$$(\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(\mathbb{I} \otimes K)(id_{\mathcal{F}} \times \varphi)(x \otimes a)] = (\pi_u^{(\frac{1}{2})} \times \pi_v \times id_{\mathcal{A}})[(id_{\mathcal{F}} \times \varphi')(x \otimes a)(\mathbb{I} \otimes K)] \quad (46)$$

for all $x \in \mathcal{F}$, $a \in \mathcal{A}$. Setting $K = x \otimes a$ and using (iii) of (40) together with (39), (38), both sides can be expanded in the spectral parameter u . In the spin- $\frac{1}{2}$ representation, setting $u \rightarrow \infty$ or $u \rightarrow 0$ it follows that the leading terms are given in terms of (44). We find that the K -matrix (13) must satisfy

$$K(v)\pi_{vq^{1/4}}^{(\frac{1}{2})}[\zeta(a)] = \pi_{v^{-1}q^{1/4}}^{(\frac{1}{2})}[\zeta(a)]K(v) \quad \text{for all} \quad a \in \{I, A, A^*\} . \quad (47)$$

This intertwining equation is actually *sufficient* to determine $K(u)$. Indeed, in this representation one has $\pi^{(\frac{1}{2})}(S_{\pm}) = \sigma_{\pm}$, $\pi^{(\frac{1}{2})}(s_3) = \sigma_3/2$. From the diagonal part of (47), it is easy to get the expressions $\Omega_0(v)$ and $\Omega_3(v)$ in (22). Replacing them in the off-diagonal part of (47), it follows that $K_{ij}(v) \in \mathcal{F}un(v; A, A^*)$ for $i \neq j$ are polynomials of degree $-2, 2, 0$ in the spectral parameter v . Also, one finds easily that the spectral parameter dependent terms must commute with A, A^* , whereas the remaining terms (denoted W, \tilde{W} in (14)) must satisfy for instance:

$$\begin{aligned}q^{1/2}A^*\tilde{W} - q^{-1/2}\tilde{W}A^* &= -c_-A(q^{1/2} + q^{-1/2}) , \\ q^{-1/2}A\tilde{W} - q^{1/2}\tilde{W}A &= c_-A^*(q^{1/2} + q^{-1/2})\end{aligned}\quad (48)$$

and similarly for W using c_+ . However, from the diagonal part of (47), one also finds $c_+(\tilde{W} - c_0W) = [A, A^*]$. Replacing in (48) one immediately gets the trilinear algebraic relations (20) together with (21). Consequently, the solution in the spin- $\frac{1}{2}$ representation (13) with (22), (20), (21) can be derived directly from (i) in (40) as soon as one knows the explicit form of the coaction acting on \mathcal{A} , or at least its asymptotic behavior in the ‘‘auxiliary’’ spectral parameter. To resume, the coaction plays here a role analogous to the coproduct in case of a Hopf algebra structure.

4 General features of the integrable structure

Either considering the constraints which follow from the reflection equation, or studying the quantum affine reflection symmetry, we have seen that the trilinear algebraic relations (20) with (21) arise in both approaches. However, it is clear that the parameters c_0, c_1, c_2 entering in the reflection matrix (13) are not restricted by any symmetry argument during this process. Instead, their explicit values will depend on the explicit realization of the Leonard pair A, A^* i.e. on *model-dependent* and *representation* characteristics of the corresponding operators. These realizations, as we will see below, admit finite/infinite dimensional or cyclic irreducible representations. Consequently, to speak in full generality i.e. without specifying a realization/representation one should instead of (20) consider the q -deformed Dolan-Grady relations (5) as the integrability condition for the class of models associated with (12).

Having identified this integrability condition, the next step is the construction of related integrable models. In the quantum inverse scattering framework, there are many examples of quantum integrable systems which are constructed starting solely from a L -operator solution of the Yang-Baxter algebra (6). Using the coproduct structure, it follows that the entries of (34) (see for instance in the Appendix) are rather nonlocal objects, usually expressed in terms of the local degrees of freedom of the system acting on the quantum space. Taking the trace over the auxiliary space of the monodromy matrix, one obtains among conserved quantities the Hamiltonian of the system. By comparison between L and K i.e. (6) and (12), we can then consider a quantum integrable model constructed solely from a K -operator of the form (13) with (22) i.e. choosing $n = 0$ in (23). Its generating function is given by

$$\tau(u) \equiv \text{Tr}_{\mathcal{V}_0} [K(u)] . \quad (49)$$

Explicitly one finds $\tau(u) = (uq^{1/2} - u^{-1}q^{-1/2})\mathbf{H}$ where the Hamiltonian⁹ takes the form (4) with $\mathbf{A} \rightarrow A$, $\mathbf{A}^* \rightarrow A^*$ which satisfy the q -deformed Dolan-Grady relations (5). Beyond the Hamiltonian, note that there exists a Casimir operator

$$Q = q\rho^* A^2 + q^{-1}\rho A^{*2} - \frac{(q - q^{-1})^2}{4} [A, A^*]_q [A, A^*]_{q^{-1}} + \frac{(q - q^{-1})}{2} \omega(AA^* + A^*A) . \quad (50)$$

An interesting problem is now to construct a family of quantum integrable models associated with (4) in the spirit of [5] based on more general objects, say \mathbf{A}, \mathbf{A}^* , satisfying (5) but *not* (20). First, by analogy with the undeformed case [4] it is indeed well expected that one can find operators \mathbf{A}, \mathbf{A}^* which admit two types of representations:

- (I) - “Single” representations i.e. \mathbf{A}, \mathbf{A}^* are represented by a pair A, A^* which obey (20);
 - (II) - “Multiple” representations (non-trivial tensor products of irreducible ones) i.e. \mathbf{A}, \mathbf{A}^* satisfy (5), *not* (20).
- Note that representations of type (II) can be obtained from more general solutions (23) of the reflection equation, as shown in [40]. Secondly, the power of the algebraic constraint (5) relies on the fact that it is an operator statement which does not refer to any local structure of the model under consideration. It is thus natural to expect that \mathbf{A}, \mathbf{A}^* are rather nonlocal objects. By analogy with the undeformed case [5], it is possible to show that there exists an (in) finite family of mutually commuting conserved quantities of the form ($r \geq 0$)

$$G^{(2r)}(\mathbf{A}, \mathbf{A}^*) = \mathbf{A}^{(2r)} + \mathbf{A}^{*(2r)} \quad \text{with} \quad G^{(0)}(\mathbf{A}, \mathbf{A}^*) \equiv \mathbf{H} \quad (51)$$

provided (5) is satisfied. For example, assuming (5) it is not difficult to check that

$$G^{(2)}(\mathbf{A}, \mathbf{A}^*) = [\mathbf{A}, [\mathbf{A}, \mathbf{A}^*]_{q^{-1}}]_q + [\mathbf{A}^*, [\mathbf{A}^*, \mathbf{A}]_{q^{-1}}]_q \quad (52)$$

is conserved. We report the detailed construction of higher conserved quantities to a separate work [40]. Then, an important problem concerns the diagonalization of these conserved quantities $G^{(2r)}(\mathbf{A}, \mathbf{A}^*)$. It is actually closely related with the problem of partial algebraization of the spectrum associated with difference equations which arises in the context of quasi-exactly solvable systems. Two situations arise:

⁹Obviously, using a scale transformation of Leonard pairs one can introduce extra parameters related with ρ, ρ^* in (4).

(1) q is root of unity: the spectrum associated with $G^{(2r)}(\mathbf{A}, \mathbf{A}^*)$ is finite. All periodic eigenfunctions can be obtained algebraically;

(2) q is not a root of unity: the spectral problem associated with $G^{(2r)}(\mathbf{A}, \mathbf{A}^*)$ is quasi-exactly solvable. It is related with the representation theory of (5). Only a part of the spectrum can be obtained, with polynomial eigenfunctions (for instance Askey-Wilson, big q -Jacobi polynomials,...) or their restrictions [26].

Using the explicit realization of the operators \mathbf{A}, \mathbf{A}^* in terms of known algebras (see below), for single representations (type (I) above) the spectral problem associated with (51) for $r \geq 0$ can be reduced to a single q -difference equation of the form

$$a(z)\Psi(qz) + d(z)\Psi(q^{-1}z) - v(z)\Psi(z) = \Lambda\Psi(z) \quad (53)$$

where $a(z)$, $d(z)$, $v(z)$ are polynomials in z defined according to the model under consideration, the realization/representation of the Leonard pair and the explicit form of $G^{(0)}$. The eigenfunctions $\Psi(z)$ of the algebraized part of the spectrum are also polynomials of finite order, Askey-Wilson polynomials for instance. Then, Λ denotes the corresponding subset of eigenvalues of the operator $G^{(0)}(\mathbf{A}, \mathbf{A}^*)$.

To be more explicit, let us give two examples corresponding to the cases (1) and (2) above and single representations (type (I)). For instance, following our results in [18] it is not difficult to find a realization for the operators \mathbf{A}, \mathbf{A}^* in terms of the Weyl algebra. Choosing

$$\mathbf{A} = -i\frac{\rho^{1/2}}{q - q^{-1}}(Q + Q^{-1}) \quad \text{and} \quad \mathbf{A}^* = -i\frac{\rho^{*1/2}}{q - q^{-1}}(P + P^{-1}) \quad \text{where} \quad PQ = q^{-1}QP, \quad (54)$$

one can check that \mathbf{A}, \mathbf{A}^* satisfy (5) and *also* (20) for $\gamma = \gamma^* = \omega = \eta = \eta^* = 0$. For $q = \exp(2i\pi M/N)$ root of unity where M, N are mutually prime integers, it is well-known that the Weyl algebra can be realized using $N \times N$ matrices with the matrix elements $\langle m|Q|n \rangle = q^{-m}\delta_{m,n}$ and $\langle m|P|n \rangle = \delta_{m+1,n \pmod N}$ for which one has $P^N = Q^N = 1$. In this case, the pair \mathbf{A}, \mathbf{A}^* admits a finite dimensional (cyclic) representation but the matrix representing \mathbf{A}^* is not tridiagonal. For these values of q , the Hamiltonian (4) together with (54) actually coincides with the Azbel-Hofstadter (AH) Hamiltonian [42] which describes the problem of Bloch electrons in a magnetic field on a two-dimensional lattice. This model has been studied in details in several works¹⁰ (see for instance [43, 44, 45, 46]). Using the relation between the group of magnetic translations and the quantum group $U_q(sl_2)$, it was shown that the AH Hamiltonian can be written as a linear combination of the quantum group generators [43]. Also, its spectrum can be represented in terms of solutions of Bethe ansatz equations on high genus algebraic curves [43, 44]. In particular, for q a root of unity the polynomial eigenfunctions and the Bethe ansatz cover all the spectrum. Previous analysis shows that the AH Hamiltonian on a square lattice in the chiral gauge can be written as a linear combination of operators of the form (54), the operators P, Q being identified with the generators of magnetic translations in each direction of the two-dimensional lattice. Using the bilinear formulation of (20) in terms of the Askey-Wilson algebra [38], it also explains the relation between the zero mode wave function and AW polynomials pointed out in [43]. Consequently, the AH Hamiltonian provides an example of quantum integrable system associated with the q -deformed Dolan-Grady relations (5).

Our second example is quasi-exactly solvable i.e. corresponds to generic values of q . For simplicity, let us consider the values $\rho = -(q - q^{-1})^2$ and $\rho^* = -abcdq^{-1}(q - q^{-1})^2$ in (5). Leonard pairs in this case admit an infinite dimensional representation in the basis of the Askey-Wilson polynomials with continuous weight¹¹ [26]. These q -orthogonal polynomials, here denoted $P_n(z)$ with integer $n \geq 0$, are symmetric Laurent polynomials of the variable z i.e. they are invariant under the change $z \rightarrow z^{-1}$. They can be written explicitly in terms of the basic q -hypergeometric functions as

$$P_n(z) = {}_4\Phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix} ; q, q \right). \quad (55)$$

¹⁰For an alternative derivation of the AH Hamiltonian starting from a L -operator instead of K , we report the reader to the Appendix.

¹¹For finite dimensional representation, it leads to Askey-Wilson polynomials with discrete weights.

In particular, these polynomials are known to satisfy the three terms recurrence relation

$$\beta_n P_{n+1}(z) + \alpha_n P_n(z) + \gamma_n P_{n-1}(z) = (z + z^{-1}) P_n(z) \quad (56)$$

where $P_{-1} \equiv 0$, $P_1 \equiv 1$ and

$$\begin{aligned} \beta_n &= \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \\ \gamma_n &= \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}, \\ \alpha_n &= a + a^{-1} - \beta_n - \gamma_n. \end{aligned}$$

In this basis and choosing the symmetric case $abcd = q$, one has

$$\pi(A) = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \cdot \end{pmatrix} \quad \pi(A^*) = \text{diag}(2, q + q^{-1}, q^2 + q^{-2}, \cdot, \cdot, \cdot) . \quad (57)$$

On the other hand, these polynomials are eigenvectors [39] of a second order q -difference operator \mathcal{D} with parameters a, b, c, d : $\mathcal{D}P_n(z) = (abcdq^{n-1} + q^{-n})P_n(z)$. The Leonard pair in this representation becomes $\pi(A) = z + z^{-1}$, $A^* = \mathcal{D}$. Setting $abcd = q$, it follows that the Askey-Wilson polynomials (55) satisfy the q -difference equation

$$\xi(z)P_n(qz) + \xi(z^{-1})P_n(q^{-1}z) = (\xi(z) + \xi(z^{-1}) - (1 - q^{-n})(1 - q^n))P_n(z) , \quad (58)$$

where

$$\xi(z) \equiv \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)} .$$

Using (56), (57) and (58), the spectral problem associated with the Hamiltonian (4) for $\rho = \rho^* = -(q - q^{-1})/2$ takes the form (53) together with

$$a(z) = \xi(z) , \quad d(z) = \xi(z^{-1}) \quad \text{and} \quad v(z) = \xi(z) + \xi(z^{-1}) - z - z^{-1} - 2 . \quad (59)$$

Using the standard Bethe ansatz technique, it follows that the roots $z_m^{(n)}$ of the Askey-Wilson polynomial (55) - eigenfunctions of (4) - determine the spectrum and obey the Bethe ansatz equations. This problem was treated in details in [41] so we don't need to repeat the analysis here and report the reader to this work for details.

Finally, it should be stressed that according to the model and realization of the Leonard pair considered, the values of the structure constants $\rho, \rho^*, \eta, \eta^*, \omega$ in (20) determine the form of polynomial eigenfunctions. Obviously, the two examples above do not exhaust all possibilities. For instance, if $\omega \neq 0$ and $\rho = \rho^* = 0$, the eigenfunctions reduce to q -Hahn polynomials associated with ${}_3\Phi_2$. For $\omega = \rho = \rho^* = 0$ they correspond to q -Krawtchouk polynomials associated with ${}_2\Phi_1$.

5 Related two-dimensional quantum integrable models

5.1 The sine-Gordon model revisited

For the coupling constant $\hat{\beta}^2 < 2$, the bulk sine-Gordon model in 1+1 Minkowski space-time is massive and integrable. The particle spectrum consists of a soliton/antisoliton pair $(\psi_+(\theta), \psi_-(\theta))$ with mass M and neutral particles, called "breathers", $B_n(\theta)$ $n = 1, 2, \dots, < \lambda$. As usual, $E = M \cosh \theta$ and $P = M \sinh \theta$ the on-shell

energy and momentum, respectively, of the soliton/antisoliton. In this model, \mathcal{H} is the Fock space of multi-particle states. A general N -particles state is generated by the “particle creation operators” $A_{a_i}(\theta_i)$

$$|A_{a_1}(\theta_1)\dots A_{a_N}(\theta_N)\rangle = A_{a_1}(\theta_1)\dots A_{a_N}(\theta_N)|0\rangle \quad (60)$$

where a_i characterizes the type of particle. The commutation relations between these operators are determined by the S -matrix elements, which describe the factorized scattering theory [47]. Integrability imposes strong constraints on the system which imply that the general S -matrix factorizes in two-particle/two-particle amplitudes which satisfy the quantum Yang-Baxter equations. To determine this basic S -matrix, an alternative approach was proposed in [48] based on the existence of non-local conserved charges in quantum integrable field theory. For the sine-Gordon model, the symmetry algebra is known to be identified with $U_{q_0}(\widehat{sl_2})$ where the relation between the deformation parameter and the coupling constant is $q_0 = \exp(-2i\pi/\hat{\beta}^2)$. To exhibit this underlying quantum symmetry, the authors of [48] used the Lagrangian representation of the sine-Gordon model:

$$\mathcal{A}_{SG} = \int d^2x \left(\frac{1}{8\pi} (\partial_\nu \phi)^2 - 2\mu \cos(\hat{\beta}\phi) \right). \quad (61)$$

Here μ is a mass scale and ϕ is the sine-Gordon field. In the perturbed conformal field theory (CFT) approach, the corresponding free bosonic fundamental field is decomposed in its holomorphic/antiholomorphic components¹²

$$\phi(x, y) = \varphi(z) + \bar{\varphi}(\bar{z}).$$

The non-local conserved currents $J_\pm(x, t), \bar{J}_\pm(x, t)$ of the form

$$J_\pm(x, t) =: \exp\left(\pm \frac{2i}{\hat{\beta}} \varphi(z)\right): \quad \text{and} \quad \bar{J}_\pm(x, t) =: \exp\left(\mp \frac{2i}{\hat{\beta}} \bar{\varphi}(\bar{z})\right):$$

generate the non-local conserved charges denoted Q_\pm, \bar{Q}_\pm below. Due to the non-locality of the currents, at equal time t they possess non-trivial braiding between themselves. Then, together with the “bulk” topological charge

$$\mathcal{T} = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi \quad (62)$$

they generate the quantum enveloping algebra $U_{q_0}(\widehat{sl_2})$ [48]. Contrary to the local integrals of motion, these conserved charges are not in involution. Their action over asymptotic soliton states can be deduced from a field theoretic approach [48]: for a single asymptotic state¹³ with topological charge $2m$

$$|v; m\rangle \quad \text{where} \quad v = \exp(\theta), \quad -j \leq m \leq j \quad (63)$$

in the spin- j representation and fixed rapidity θ it reads

$$\pi_v[Q_\pm] = ce^{\lambda\theta} S_\pm q_0^{\pm s_3}, \quad \pi_v[\bar{Q}_\pm] = ce^{-\lambda\theta} S_\pm q_0^{\mp s_3} \quad \text{and} \quad \pi_v[\mathcal{T}] = 2s_3 \quad (64)$$

where

$$\lambda = \frac{2}{\hat{\beta}^2} - 1 \quad \text{and} \quad c^2 = -i \frac{\mu}{\lambda^2} (q_0^{-2} - 1).$$

More generally, the action of the non-local conserved charges Q_\pm, \bar{Q}_\pm and \mathcal{T} over a multi-particle state is obtained using the coproduct rules of $U_{q_0}(\widehat{sl_2})$ (see [48] for details). In the following, we will associate one-particle states to single representations (type (I)) and multi-particle states to representations of type (II).

¹²If we denote the expectation value over the Fock space vacuum of the CFT by $\langle \dots \rangle$, then these components are normalized such that $\langle \varphi(z) \varphi(w) \rangle = -\ln(z-w)$, $\langle \bar{\varphi}(\bar{z}) \bar{\varphi}(\bar{w}) \rangle = -\ln(\bar{z}-\bar{w})$, $\langle \varphi(z) \bar{\varphi}(\bar{w}) \rangle = 0$.

¹³For the fundamental soliton/antisoliton with topological charge ± 1 , $j = 1/2$. Higher values of j occur for boundstates.

5.1.1 Askey-Wilson dynamical symmetry and duality

A natural question is whether certain combinations of non-local conserved charges might satisfy the relations (5) and generate conserved quantities in *involution*. In order to exhibit such non-trivial combinations in the sine-Gordon model, we use the following trick. According to the analysis of Section 2 and (23), the simplest (up to an overall factor) non-trivial “dynamical” solution of the RE (12) is of the form (13) with (22). An easy way to find single representations (type (I)) of operators satisfying (5) is then to start from a non-dynamical solution $K^{(0)}(u)$, even very simple, and to apply (23) for $n = 1$ using the Lax operator (8). For instance, let us choose the simple off-diagonal constant solution of (12) given by $K \equiv (c_+ \sigma_+ + c_- \sigma_-)/(q^{1/2} - q^{-1/2})$ and set $v \rightarrow v_0 q^{-1/4}$ in (38). It is straightforward to obtain the solution (13) with (22) through the identification

$$A \equiv c_+ v_0 S_+ q^{s_3/2} + c_- v_0^{-1} S_- q^{s_3/2} \quad \text{and} \quad A^* \equiv c_- v_0 S_- q^{-s_3/2} + c_+ v_0^{-1} S_+ q^{-s_3/2} . \quad (65)$$

Here we use (45), (20) with (21) and we find the relation

$$c_2(v_0; j) = \frac{(v_0^2 q^{-1/2} + v_0^{-2} q^{1/2}) w_j}{c_0 c_1 (q^{1/2} + q^{-1/2})(q - q^{-1})} \quad (66)$$

where w_j is the Casimir operator (11) eigenvalue

$$w_j = q^{j+1/2} + q^{-j-1/2} . \quad (67)$$

An other useful realization of operators satisfying (5) can be obtained from the non-diagonal non-dynamical solution of the reflection equation¹⁴[50], [16]. It yields to

$$A \equiv c_+ v_0 S_+ q^{s_3/2} + c_- v_0^{-1} S_- q^{s_3/2} + \epsilon_+ q^{s_3} \quad \text{and} \quad A^* \equiv c_- v_0 S_- q^{-s_3/2} + c_+ v_0^{-1} S_+ q^{-s_3/2} + \epsilon_- q^{-s_3} \quad (68)$$

where ϵ_{\pm} are free parameters. Actually, it is not difficult to check that the above simple realizations of A, A^* satisfy (20) with $\gamma = \gamma^* = 0$, $\rho = \rho^* = 1/c_0 c_1^2$ and the structure constants

$$\begin{aligned} \omega(v_0; j) &= -\frac{c_2(v_0; j) + \epsilon_+ \epsilon_- c_1 (q^{1/2} - q^{-1/2})}{c_1} (q^{1/2} - q^{-1/2}) , \\ \eta(v_0; j) &= \frac{\epsilon_+ (v_0^2 q^{-1/2} + v_0^{-2} q^{1/2}) - \epsilon_- w_j}{c_0 c_1^2 (q^{1/2} + q^{-1/2})} , \\ \eta^*(v_0; j) &= \eta(v_0; j)|_{\epsilon_+ \leftrightarrow \epsilon_-} . \end{aligned} \quad (69)$$

Here, c_0, c_1 are non-vanishing parameters, c_2 is the same as above (66) and w_j is the Casimir eigenvalue (11). It is now straightforward to identify the operators satisfying (5) in the sine-Gordon model. First, setting

$$v_0 = \exp(\lambda \theta) , \quad q_0^2 = q , \quad c_+ c_0 = c_- = -\frac{1}{c_1 (q^{1/2} + q^{-1/2})} = c$$

and using (64) one can interpret (68) as combinations of non-local charges acting on a single (one-particle state) representation (63) i.e. of type (I). In particular, the fact that (68) satisfy the trilinear algebraic relations (20) with (69) implies (5). More generally, in order to find A, A^* which do not reduce to a Leonard pair, one has to consider the action of the non-local conserved charges over multi-particle states (type (II)). In this case, we propose to consider the pair

$$\hat{Q}_+ = \frac{1}{c_0} Q_+ + \bar{Q}_- + \epsilon_+ q_0^T \quad \text{and} \quad \hat{Q}_- = Q_- + \frac{1}{c_0} \bar{Q}_+ + \epsilon_- q_0^{-T} . \quad (70)$$

To show that q -deformed Dolan-Grady relations are satisfied for *any* N -particles state representation of (70) it is necessary and sufficient to show that the coproduct of $U_{q_0}(\widehat{sl}_2)$ leaves the relations (5) unchanged. Thus, we

¹⁴For a similar calculation, see [49]. Note that this non-dynamical solution can also be obtained from (22) for A, A^* constants.

have checked explicitly that the non-local conserved charges satisfy the q -deformed Dolan-Grady relations (5) with the substitution

$$A \rightarrow \hat{Q}_+ \quad \text{and} \quad A^* \rightarrow \hat{Q}_- \quad (71)$$

for any $c_0 \neq 0$ and free parameters ϵ_{\pm} (for details, see [40]). In this case, the value of ρ , ρ^* in (5) is

$$\rho = \rho^* = \frac{(1+q)(1+q^{-1})}{c_0} . \quad (72)$$

We would like to stress that the trilinear algebraic relations (20) only arise in case of a single representation (one-particle state). This algebra actually admits a bilinear formulation in terms of the Askey-Wilson algebra AW(3) introduced and studied by Zhedanov in [51]. It is generated by the elements $\{K_0, K_1, K_2\}$ which satisfy

$$[K_0, K_1]_q = K_2 , \quad [K_2, K_0]_q = BK_0 + C_1K_1 + D_1I , \quad [K_1, K_2]_q = BK_1 + C_0K_0 + D_0I$$

where B, C_0, C_1, D_0, D_1 are the structure constants. It is an exercise to relate these structure constants to the ones in (69). According to the analysis of [51], one can obtain irreducible finite dimensional representations depending on the value of the rapidity and the dimension of the asymptotic state representation $2j+1$. Following [51], it is possible to find a basis $|\psi_m\rangle$, $-j \leq m \leq j$, for the asymptotic particle state of rapidity θ which diagonalizes $\pi_v[\hat{Q}_+]$. On the other hand, one can find a basis $|\varphi_s\rangle$, $-j \leq s \leq j$ for the asymptotic particle state which instead diagonalizes $\pi_v[\hat{Q}_-]$. Consequently, the conserved quantities (70) enjoy a remarkable duality property which reminds the one pointed out by Dolan-Grady in their paper [5]. Since both representations have the same dimension, both basis are related by a linear transformation $|\varphi_s\rangle = \sum \langle s|p\rangle |\psi_p\rangle$. As shown in [51], the corresponding overlap functions $\langle s|p\rangle$ can be explicitly expressed in terms of the Askey-Wilson polynomials on a finite interval of the real axis.

5.1.2 Related spectral problem and asymptotic one-particle states

It is now possible to construct an infinite set of mutually commuting conserved quantities of the form (51) in the sine-Gordon model (for a detailed analysis of a slightly more general integrable structure, see [40]), the first ones being

$$\begin{aligned} H &= \hat{Q}_+ + \hat{Q}_- , \\ G^{(2)} &= [\hat{Q}_+, [\hat{Q}_+, \hat{Q}_-]_{q^{-1}}]_q + [\hat{Q}_-, [\hat{Q}_-, \hat{Q}_+]_{q^{-1}}]_q . \end{aligned} \quad (73)$$

In general, the corresponding spectral problem is rather complicated. However, acting on a one-particle state all expressions drastically simplify. The corresponding spectral problem for a state of rapidity θ and topological charge $2m$ in a representation of dimension $2j+1$ reads for $r = 0, 1, \dots$

$$G^{(2r)}|v; m\rangle = \Lambda_{2r}(v; j, m)|v; m\rangle \quad \text{for all} \quad -j \leq m \leq j . \quad (74)$$

Up to an overall (v, j) -dependent coefficient which changes according to the value of r , this spectral problem always reduces to the one for H . Indeed, using the trilinear algebraic relations (20) it is not difficult to check that if $|v; m\rangle$ is an eigenstate of H , so it will be for all higher conserved quantities. For general values of q_0 , it is a quasi-exactly solvable problem. To study it, we follow the analysis of [41]. Expressed in the weight basis, the generators of $U_{q_0}(sl_2)$ leave invariant the linear space (of $2j+1$ dimension) of polynomials $F(z)$ of degree $2j$. The lowest/highest weights are identified with $F_0 = 1$ and $F_{2j} = z^{2j}$, respectively. One has

$$\begin{aligned} q_0^{\pm s_3} F(z) &= q_0^{\mp j} F(q_0^{\pm 1} z) , \\ S_+ F(z) &= -\frac{z}{(q_0 - q_0^{-1})} (q_0^{-2j} F(q_0 z) - q_0^{2j} F(q_0^{-1} z)) , \\ S_- F(z) &= \frac{1}{z(q_0 - q_0^{-1})} (F(q_0 z) - F(q_0^{-1} z)) . \end{aligned}$$

In this basis of polynomials, for general values¹⁵ of the parameters we may represent a one-particle state with rapidity θ (recall that $v = \exp(\theta)$) by

$$\langle z|v; m\rangle = \Psi_m(z) \quad \text{with} \quad \Psi_m(z) = \prod_{n=1}^{2j} (z - z_n^{(m)}) \quad (75)$$

where $z_n^{(m)}$ denote the roots of the m -th polynomial. From (74) and using the representation above, it is straightforward to obtain a q -difference equation of the form (53) with the substitution $q \rightarrow q2_0$ where the coefficients are explicitly given by

$$\begin{aligned} a(z) &= (-c_+ v z q_0^{-3j} + c_- v^{-1} z^{-1} q_0^{-j} + \epsilon_+ (q_0 - q_0^{-1}) q_0^{-2j}) / (q_0 - q_0^{-1}) , \\ d(z) &= (-c_- v z^{-1} q_0^j + c_+ v^{-1} z q_0^{3j} + \epsilon_- (q_0 - q_0^{-1}) q_0^{2j}) / (q_0 - q_0^{-1}) , \\ v(z) &= (-c_+ v z q_0^j + c_- v^{-1} z^{-1} q_0^{-j} + c_+ v^{-1} z q_0^{-j} - c_- v z^{-1} q_0^j) / (q_0 - q_0^{-1}) . \end{aligned} \quad (76)$$

The corresponding eigenvalue $\Lambda_0(v; j, m)$ can be deduced from the results of [41]. It is expressed in terms of the roots $z_n^{(m)}$ in (75) which have to satisfy a system of $2j$ algebraic (Bethe) equations. This system has exactly $2j + 1$ solutions corresponding to $2j + 1$ eigenfunctions $\Psi_m(z)$, $-j \leq m \leq j$. Note that these roots depend on value of the rapidity θ and ϵ_{\pm} .

To resume, the sine-Gordon model possesses an underlying dynamical symmetry associated with (5) with (71). For single representations, it is isomorphic to the quadratic Askey-Wilson algebra, which is generated by the charges. This symmetry is preserved at all orders in perturbed CFT framework because the fundamental non-local charges $Q_{\pm}, \overline{Q}_{\pm}$ and \mathcal{T} are conserved. The point is that these charges generate an infinite number of mutually commuting conserved quantities $G^{(2r)}$ [40]. Actually, their actions on asymptotic states mix non-trivially the holomorphic and antiholomorphic sectors (in the perturbed CFT approach). Then, conservation of $G^{(2r)}$ for any r imposes a severe restriction on the asymptotic states structure. For instance, a one-particle state (75) admits a representation in terms of polynomials whose roots satisfy Bethe equations. The corresponding eigenvalues can be computed using the usual Bethe ansatz techniques (see for instance [41]).

5.2 The sine-Gordon model with/without a dynamical boundary

Following [9], boundary integrable models on the lattice - for instance the XXZ spin chain - can be obtained using (23). In [17] we have applied Sklyanin's formalism [9] to construct a classical integrable field theory restricted to the half-line, namely the sine-Gordon model, coupled with a mechanical system at the boundary. In general, such two-dimensional classical field theory will remain integrable provided there exists a classical solution $K_{cl}(u)$ of the classical reflection equation. In this case, the generating function reads

$$\tau_{cl}(u) \equiv \text{Tr}_{\mathcal{V}_0} [T_{cl}(u) K_{cl}(u) \hat{T}_{cl}(u^{-1})] \quad (77)$$

where $T_{cl}(u)$ (and similarly $\hat{T}_{cl}(u)$) is the sine-Gordon classical monodromy matrix. Expanding in the spectral parameter u one obtains an infinite number of mutually ‘‘Poisson’’ commuting quantities, which give the integrals of motion and ensure the integrability of the classical system with a boundary. However, it is clear that the analysis of [17] for the classical boundary sine-Gordon model can be extended to more general boundary operators provided they satisfy some classical quadratic algebra. This can be done starting from the classical limit of the reflection matrix (13). Expanding $\tau_{cl}(u)$ in the spectral parameter u , among the integrals of motion one identifies the Hamiltonian. Its boundary contribution is found to be linear in the classical boundary operators coupled with the sine-Gordon classical field ϕ . Setting the coupling constant of the sine-Gordon model $\hat{\beta} = 0$, this boundary term reduces to the classical analogue of (4).

At quantum level, an integrable (massive) boundary sine-Gordon model coupled with a quantum mechanical system at the boundary was proposed in [18]. There, the asymptotic behavior of the boundary operators was

¹⁵For some special values of the parameters, the degree of the polynomial eigenfunctions can be different. Here we do not consider this possibility.

expressed in terms of the operators (54). However, as we have shown in Sections 2 and 3, the existence of a K -matrix is related with (20) (see also [18] for details). It is thus more general to consider in full generality boundary operators which asymptotically satisfy (20) rather than choosing a (somehow simpler) realization like (54). Then, we propose

$$\mathcal{A}_{bsg} = \int_{-\infty}^{\infty} dy \int_{-\infty}^0 dx \left(\frac{1}{8\pi} (\partial_\nu \phi)^2 - 2\mu \cos(\hat{\beta}\phi) \right) + \mu^{1/2} \int_{-\infty}^{\infty} dy \Phi_{pert}^B(y) + \mathcal{A}_{boundary} \quad (78)$$

where the interaction between the sine-Gordon field and the boundary quantum operators reads

$$\Phi_{pert}^B(y) = \mathcal{E}_-(y) e^{i\hat{\beta}\phi(0,y)/2} + \mathcal{E}_+(y) e^{-i\hat{\beta}\phi(0,y)/2}$$

and $\mathcal{A}_{boundary}$ is the kinetic part¹⁶ associated with the boundary operators $\hat{\mathcal{E}}_\pm(y)$. In the deep UV, this model can be considered as a relevant perturbation of a conformal field theory (for instance the free Gaussian field) on the semi-infinite plane with dynamical boundary conditions at $x = 0$. In this case, similarly to the bulk (62) the free Gaussian field restricted to the half-line can be also written in terms of its holomorphic/anti-holomorphic components. However, these components are instead normalized such that

$$\langle \varphi(z) \varphi(w) \rangle_0 = -2 \ln(z - w), \quad \langle \bar{\varphi}(\bar{z}) \bar{\varphi}(\bar{w}) \rangle_0 = -2 \ln(\bar{z} - \bar{w}), \quad \langle \varphi(z) \bar{\varphi}(\bar{w}) \rangle_0 = -2 \ln(z - \bar{w})$$

where $\langle \dots \rangle_0$ now denotes the expectation value in the boundary conformal field theory (BCFT) with Neumann boundary conditions. The boundary conditions in (78) can be derived from the expectation value of the local field $\partial_x \phi(0, y)$ with any other local field in first order of boundary conformal perturbation theory $\mu \rightarrow 0$. Actually, these quantum boundary conditions take a form similar to the classical ones [18].

We can choose the y -direction to be the “time” in which case the Hamiltonian contains the boundary contribution, and the Hilbert space \mathcal{H}_B is identified with the half-line $y = Const.$. Then, correlation functions are calculated over the boundary ground state denoted $|0\rangle_B$ below. This state can be expanded as

$$|0\rangle_B = |vac\rangle_B \otimes |0\rangle_{BCFT} + \mathcal{O}(\mu), \quad (79)$$

where $|0\rangle_{BCFT} \in \mathcal{H}_{BCFT}$ belongs to the Hilbert space of the BCFT and $|vac\rangle_B$ is an eigenstate of the boundary Hamiltonian for the coupling $\hat{\beta} \rightarrow 0$.

There are good reason to believe [16, 18] that a quantum analogue of the classical integrals of motion can be constructed explicitly. Similarly to the bulk case, instead one can show that the model (78) remains integrable at quantum level *too* using the existence of *boundary* non-local conserved charges. Along the line of [30, 32, 18] it is possible to construct non-local conserved charges in (78) in terms of the bulk ones (which are no longer conserved individually). Assuming the same asymptotic behavior at the end point of the time axis

$$\hat{\mathcal{E}}_+(y = \pm\infty) \sim \hat{\mathcal{E}}_+ \quad \text{and} \quad \hat{\mathcal{E}}_-(y = \pm\infty) \sim \hat{\mathcal{E}}_-, \quad (80)$$

it follows that the quantum affine reflection symmetry of the model (78) is generated by the operators

$$\hat{Q}_\pm^{(0)} = Q_\pm + \bar{Q}_\mp + \hat{\mathcal{E}}_\pm q_0^\mp \mathcal{T}_b \quad \text{with} \quad \mathcal{T}_b = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^0 dx \partial_x \phi \quad (81)$$

and $\hat{\mathcal{E}}_\pm(y) = \mu^{1/2} \mathcal{E}_\pm(y) \hat{\beta}^2 / (1 - \hat{\beta}^2)$. For the special case (54), this was shown in details in [18]. Here, \mathcal{T}_b denotes the boundary topological charge which is no longer independently conserved. In the following, we are going to consider different restrictions of the boundary operators. This will reveal interesting features of the model (78). In particular, most of the known and well-studied massive or massless boundary integrable models can be seen as limiting cases of (78).

• **The boundary sine-Gordon model revisited.** The quantum boundary sine-Gordon model introduced by Ghoshal-Zamolodchikov is obtained for the one-dimensional (trivial) representation of the boundary operators

¹⁶A solution of the “dual” reflection (associated with a boundary on the left hand side) can be used to introduce such term, without breaking integrability.

(which become free parameters) in (78). In other words, the boundary degrees of freedom are “frozen” to constants. The corresponding non-local conserved charges calculated in [30, 32] take the form (81) together with the substitution $\hat{\mathcal{E}}_{\pm} \rightarrow \epsilon_{\pm}$, where ϵ_{\pm} are free parameters¹⁷. It is then interesting to notice that, from an algebraic point of view, they coincide exactly with the spin- $(j = \frac{1}{2})$ representation of (70) together with

$$c_0 = 1, \quad \mathcal{T}_b \rightarrow -\mathcal{T}.$$

This should not be surprising as the symmetry between the holomorphic/antiholomorphic sector must be restored at the boundary. Due to the presence of the boundary which breaks the $U_{q_0}(\widehat{sl}_2)$ symmetry, the quantum group symmetry of the boundary sine-Gordon model with fixed boundary conditions is generated by the non-local conserved charges (70) for the special case (symmetric) $c_0 = 1$. In particular, single representations are associated with a restricted Askey-Wilson quadratic algebra. As before, the non-local boundary conserved charges satisfy the q -deformed Dolan-Grady relations (5) with (69) and $c_0 = 1$. Conservation of the quantities $G^{(2r)}$ for $c_0 = 1$ implies that the boundary space representation has dimension $2j + 1$. A convenient basis for the boundary (degenerate) ground state $|0\rangle_B$ is provided by the Askey-Wilson polynomials with discrete weights, or can be written in the same manner than (75).

The boundary scattering properties are known to be encoded in the boundary reflection matrix, which can be obtained using the quantum reflection symmetry [30] (see also [32]). In the formalism of Section 3, it corresponds to the choice of representation $\pi^{(1/2)} \times \pi^{(0)}$ in (44) where $\pi^{(0)}(A) \sim \epsilon_+$, $\pi^{(0)}(A^*) \sim \epsilon_-$ for the non-local charges. Using the convenient parameterization of [53], it is straightforward to relate (13) with the reflection matrix of [16] using the appropriate unitarization factor.

• **Massless limit: anisotropic Kondo and Bazhanov-Lukyanov-Zamolodchikov models.** The massless limit of the model (78) can be reached in various ways, according to the realization of the boundary operators¹⁸ one would like to have. In the dynamical boundary case, it should be reminded that the integrability is preserved for boundary operators which satisfy asymptotically (20) and the substitution $A \rightarrow \hat{\mathcal{E}}_+$, $A^* \rightarrow \hat{\mathcal{E}}_-$ with $\rho = \rho^* = c^2(q^{1/2} - q^{-1/2})$ (see [18] for details). The “massless” limit of (78) is obtained if the eigenvalues of $\mu^{1/2}\hat{\mathcal{E}}_{\pm}|_{\mu \rightarrow 0}$ are finite. This can be done, for instance, by taking the limit $v_0 \rightarrow \infty$ in (68) such that $v_0^2\mu \sim 1$. It gives

$$A \rightarrow c_+ v_0 S_+ q^{s_3/2} \quad \text{and} \quad A^* \rightarrow c_- v_0 S_- q^{-s_3/2} \quad (82)$$

where (45). Using (82) in (78), the resulting Hamiltonian becomes

$$H_{massless} \sim \frac{1}{8\pi} \int_{-\infty}^0 dx \left((\pi(x))^2 + (\partial_x \phi(x))^2 \right) - \mu_B^{1/2} (S_+ q^{s_3/2} e^{i\hat{\beta}\phi(0)/2} + S_- q^{-s_3/2} e^{-i\hat{\beta}\phi(0)/2}) \quad (83)$$

For $q = 1$, this Hamiltonian is the bozonized version of the anisotropic Kondo model [54]. For $q \neq 1$, one obtains the spin- j generalization studied in details in [55].

An other example of massless model is obtained using a different realization of the Lax operator in (23), and proceeding by analogy to obtain (65). Taking the appropriate limit, one obtains

$$A \rightarrow i(q^{1/2} - q^{-1/2})^{1/2} a_+ \quad \text{and} \quad A^* \rightarrow i(q^{1/2} - q^{-1/2})^{1/2} a_- q^{N/2}, \quad (84)$$

where the operators a_+, a_- and q^N generate the q -oscillator algebra

$$a_- a_+ - q^{\pm 1} a_+ a_- = q^{\mp N} \quad \text{and} \quad q^N a_{\pm} q^{-N} = q^{\pm 1} a_{\pm}. \quad (85)$$

Notice that $q^{1/2} A A^* - q^{-1/2} A^* A = q^{1/2} - q^{-1/2}$. For this realization of the boundary operators (in which case (5) simplify to q -Serre relations), the model (78) reduces to the one proposed by Bazhanov-Lukyanov-Zamolodchikov in [20] at zero voltage¹⁹. This model finds interesting application in non-equilibrium systems.

¹⁷These parameters are expressed in terms of the boundary parameters in the Lagrangian and some bulk contributions. I thank Z. Bajnok who attracted my attention to this point.

¹⁸I thank H. Saleur for stimulating communications about this problem.

¹⁹In case of non-zero voltage, one can use a slightly different expression for the generating function.

• **Reflectionless points: Bassi-LeClair massive Kondo model.** The reflectionless points correspond to the values of the coupling constant $\hat{\beta}$ is such that $q = \pm 1$. In [56] a rather general massive version of the anisotropic spin 1/2 Kondo model has been proposed. At the reflectionless points, it was claimed to be integrable unlike the massless version which remains integrable for arbitrary values of $\hat{\beta}$. Using (65), one can see that the Hamiltonian associated with (78) coincides with the model proposed by Bassi and LeClair in [56] for $q = 1$. A more interesting property can furthermore be exhibited: at this value of q , the boundary operators $\hat{\mathcal{E}}_{\pm}$ and boundary non-local conserved charges $\hat{Q}_{\pm}^{(0)}$ generate the infinite dimensional Onsager algebra (2). It is then possible to construct the corresponding conserved quantities in involution which ensure the integrability of the theory and confirms the conjecture of [56]. Furthermore, this might open a way to extract exact information (correlation functions,...) in the massive regime.

• **Remark on the decoupling limit: Bulk/Boundary “dual” description.** It is rather instructive to consider the limit in which the bulk degrees of freedom (sine-Gordon field) and the boundary ones $\hat{\mathcal{E}}_{\pm}(y)$ decouple. This can be realized either for Neumann boundary conditions, or vanishing coupling constant $\hat{\beta} = 0$ in (78). For Neumann boundary conditions, i.e. $\hat{\mathcal{E}}_{\pm}(y) = 0$, only the bulk sine-Gordon contribution and the term $\mathcal{A}_{boundary}$ survive. The non-local conserved charges in this limit [32] simplify to $\hat{Q}_{\pm}^{(0)}|_{\hat{\mathcal{E}}_{\pm} \rightarrow Const.}$ in (81). On the other hand, in the second case $\hat{\beta} = 0$, the action (78) becomes

$$\mathcal{A}_{bsg}|_{\hat{\beta}=0} = \int_{-\infty}^{\infty} dy (\mathcal{E}_{-}(y) + \mathcal{E}_{+}(y)) + \mathcal{A}_{boundary} . \quad (86)$$

In the asymptotic limit $(x, y) \rightarrow \pm\infty$, it is easy to check that $\hat{\mathcal{E}}_{\pm}$ (for $\hat{\beta} = 0$) and $\hat{Q}_{\pm}^{(0)}|_{\hat{\mathcal{E}}_{\pm} \rightarrow Const.}$ (for Neumann boundary conditions) satisfy exactly the *same* algebraic relations (5). In view of the weak-strong coupling duality, in the perturbative regime $\hat{\beta} \rightarrow 0$ in (78), one can see that the boundary contribution (86) actually contains some informations about the nonperturbative regime which is usually characterized by the non-local conserved charges. This might explain why, for the analytic continuation $\hat{\beta} \rightarrow ib$ in (78), the dynamical boundary sinh-Gordon model is self-dual under weak-strong coupling duality (the reflection amplitude of the fundamental particle is invariant under the change $b \leftrightarrow 2/b$) [18], contrary to the model with fixed boundary conditions [53].

6 Concluding remarks

The spectral parameter dependent reflection equation is usually considered in the context of integrable systems with boundaries. However, based on the analysis of this quadratic algebra we have exhibited a new quantum integrable structure which is not specific to boundary systems. The integrability condition was shown to be associated with q -deformed Dolan-Grady relations (5). Our construction, which extends the one proposed by Dolan-Grady some years ago [5], leads to the existence of an (in)finite set of conserved quantities in involution [40]. In particular, it shows that the concept of “superintegrability” can be further extended for $q \neq 1$. In this direction, it is a challenge to find the corresponding (in)finite dimensional q -Onsager algebra that generate these conserved quantities in massive theories, and would provide an alternative approach to massive quantum integrable models. This new hidden symmetry (5) was exhibited in various bulk or boundary quantum integrable models in which case the operators \mathbf{A}, \mathbf{A}^* have been identified and the first conserved quantities (4) and (52) constructed explicitly. Among the interesting applications, we have shown that the sine-Gordon model enjoys a remarkable dynamical symmetry associated with the quadratic Askey-Wilson algebra. From this, we have obtained the asymptotic one-particle states in terms of q -orthogonal polynomials, as well as the structure of the boundary space of state in this model with a boundary. From a more general point of view, we have also shown that several examples of massive/massless boundary integrable models are special cases of the model (78). Finally, we have pointed out a bulk-boundary “dual” relation in the sine-Gordon model with a dynamical boundary. Although we didn’t discuss the XXZ spin chain with/without a boundary here, a similar description also holds in this model.

An interesting problem is whether the present approach can be generalized to higher rank affine Lie algebras. Actually, there has been some interest in generalizations of the Onsager algebra and corresponding Dolan-Grady

relations. Indeed, the Onsager algebra is intimately related with sl_2 . In case of sl_n , such extension was studied in details in [58]. There, an integrable Hamiltonian of the form

$$H = \kappa_1 A_1 + \kappa_2 A_2 + \dots + \kappa_n A_n, \quad n \geq 3 \quad (87)$$

was proposed where κ_i are arbitrary constants. This Hamiltonian is a member of an infinite family of commuting integrals of motion if the operators A_i satisfy certain trilinear relations. From our results, it is tempting to consider a higher rank q -deformed generalization of (5) with the following trilinear q -deformed relations:

$$[A_i, [A_i, A_j]_q]_{q^{-1}} = \rho_i A_j, \quad \text{and} \quad [A_i, A_j] = 0 \quad \text{if} \quad |i - j| > 1 \quad (88)$$

where $\{i, j\} \in \{1, \dots, n\}$. In some sense, these relations extend the concept of Leonard pair to any finite Lie algebra. Obviously, using an appropriate shift $A_i \rightarrow A_i + \alpha_i$ it is not difficult to generalize the relations (20). For the same reasons as before, it is consequently sufficient to focus on the relations (88). For $q = 1$ in (88), one obtains the relations found in [58]. Also, for $\rho_i = 0$, one recovers the q -Serre relations associated with $U_q(sl_n)$. We intend to consider this model together with applications to massive Toda field theories in a separate publication.

To conclude, we would like to stress that the power of the relations (5) lie in the algebraic statement which does not refer to the number of dimensions or the nature of the space-time (continuous, discrete,...) “hidden” behind the operators A, A^* . An interesting problem is consequently to find some examples where such hidden symmetry appears explicitly in higher dimensions.

Note added:

Using the realization (54), it is easy to relate our general expression (13) with (22) to known results. For instance, choosing $c_0 = \mp 1$, $c_1 = \mp 2/(q - q^{-1})$, $c_2 = 0$ and $q \rightarrow q^{-2}$ it is not difficult to check that (22) coincides with the one proposed in [59] (for $(-)$) (see also [17]) or the one that follows from [18] (for $(+)$).

Also, it should also be noted that the reflection matrix (13), for some special realizations of Leonard pairs in terms of $U_{q^{1/2}}(\widehat{sl_2})$ can be related with a solution proposed some years ago by A. Zabrodin in [49]. This solution was obtained from the non-dynamical K -matrix in [16] and L -operators satisfying (32). It should be stressed that our solution (13) is derived independently of a realization of Leonard pairs, either directly from the RE (12) or using the quantum affine reflection symmetry described in Section 3.

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Appendix

In this appendix, we want to show that an Hamiltonian of the form (4) can be obtained starting from an L -operator instead of a K -operator. Indeed, in the quantum inverse scattering method, an integrable lattice theory can be constructed considering the monodromy matrix

$$T(u) = L_{\mathcal{V}_0 \mathbf{N}}(u) \dots L_{\mathcal{V}_0 2}(u) L_{\mathcal{V}_0 1}(u) \quad (89)$$

where $L_{\mathcal{V}_0 j}(u)$ acting on the quantum space \mathcal{V}_j is a solution of the Yang-Baxter algebra (6) for $\mathcal{V}_0 = \mathcal{V}'_0$ and $\mathcal{V} \rightarrow \mathcal{V}_j$. Taking the trace over the auxiliary space \mathcal{V}_0 of (89), one obtains the transfer matrix

$$\tau(u) = \text{Tr}_{\mathcal{V}_0}(T(u)) \quad (90)$$

acting on $\otimes_{j=1}^N \mathcal{V}_j$. This generating function provides a set of mutually commuting conserved quantities. For instance, let us focus on the Yang-Baxter algebra (6) with (7) associated with $U_{q^{1/2}}(\widehat{sl_2})$ in the spin- $\frac{1}{2}$ auxiliary space representation $\mathcal{V}_0 = \mathcal{V}'_0$ and $N = 1$ in (89). Following [43], one can consider

$$L(u) = \begin{pmatrix} -i\kappa^* \rho^{*1/2} P - i\kappa \rho^{1/2} Q & ukq^{-1/4} (PQ^{-1})^{1/2} - u^{-1} k^{-1} q^{1/4} (PQ^{-1})^{-1/2} \\ ukq^{1/4} (PQ^{-1})^{-1/2} - u^{-1} k^{-1} q^{-1/4} (PQ^{-1})^{1/2} & -i\kappa \rho^{1/2} Q^{-1} - i\kappa^* \rho^{*1/2} P^{-1} \end{pmatrix} \quad (91)$$

with parameters $\{k, \kappa, \kappa^*\} \in \mathcal{C}$. Using the realization of the Leonard pair A, A^* in terms of the Weyl algebra \mathcal{W}_q (54), one obtains from (90) a Hamiltonian acting on \mathcal{V} of the form

$$H = (q - q^{-1})(\kappa A + \kappa^* A^*) \quad (92)$$

where A, A^* satisfy the q -deformed Dolan-Grady relations (5).

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